Oscillations of tide and surge in an estuary of finite length

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SUMMARY

This paper concerns a narrow basin of uniform cross-section open to the sea at one end and closed at the other. An incident long wave of prescribed general form is supposed to enter from the sea and to represent the combination of tide and surge as generated in the sea. The solution of the linear terms of the equations of continuity and motion gives the reflection of this wave at the head of the estuary. This paper gives the next approximation when the non-linear terms are retained, the second-order motion being made determinate by the condition that, at the mouth, it reduces to a progression towards the sea.

The chief results relate to the surface elevation at the head of the estuary. When the first order elevation there increases steadily to a maximum, the effect of the 'shallow water terms' is to make high water higher and earlier, while the effect of the 'frictional term' is to make high water lower and later. For a short estuary, the interaction of the tide on a surge, due to a given sequence of meteorological conditions over the sea, is to make it higher when its maximum occurs at the time of tidal high water than when its maximum occurs at the time of tidal low water. This is directly opposite to the corresponding result when the estuary is of infinite length.

1. Introduction

In two papers (Proudman 1955 a, b) I have discussed the dynamics of a progressive wave of tide and surge in an estuary, and particularly the interaction between the tide and the surge. For equal sequences of meteorological conditions over the sea, I showed that the apparent height of a surge whose maximum occurs near to the time of tidal high water is less than that of a surge at the same place whose maximum occurs near to the time of tidal low water. But the tides of the Thames Estuary, for example, constitute a standing oscillation much more nearly than they constitute a progressive wave.

In the first of the papers, I also gave an approximate solution of the differential equations relating to an estuary with a barrier, but without using end-conditions which would completely determine the motion. In 1956 Doodson considered the problem presented by a gulf, and gave numerical solutions for the cases of a tide prescribed at the mouth and of a surge prescribed at the head. Because of reflection from the head, it is inappropriate to prescribe a surge at the mouth of an estuary. Also, any attempt to deal with discontinuities of basin at the mouth, involves reflections from the mouth, both on the seaward side and on the estuarial side.

In this paper, a prescribed incident wave is taken to progress up the estuary, and the remaining part of the motion is taken to reduce, at the mouth, to a wave travelling down the estuary. The incident wave is taken to represent the combined tide and surge which are generated in the sea, and the remainder of the motion is taken to represent the origin of the wave which, in the open sea, will diverge away from the mouth of the estuary.

In order to avoid a greater mathematical complication, I suppose the cross-section of the estuary to be uniform. I follow the same mathematical method as in my earlier paper, except that I now use the differential equation for the current, whereas previously I used that for the elevation of the water-surface. Because of the more precise end-conditions of the present paper, the results are more definite than those of that part of the earlier paper which relates to an estuary with a barrier.

It appears that the surface-elevation at the head of the estuary depends on the primary elevation there at the same time and for a previous interval during which a progressive wave could travel twice the length of the estuary. When, during this interval, the primary wave increases steadily to a maximum, the tendency of the shallow water terms of the differential equations is to make high water higher and earlier, while the tendency of the frictional term is to make high water lower and later. For a short estuary, the current in the estuary, and the time and height of high water at its head, are approximately independent of friction; while the interaction of the tide on a surge, due to a given sequence of meteorological conditions over the sea, is to make it higher when its maximum occurs at the time of tidal high water than when its maximum occurs at the time of tidal low water. This result is directly opposite to the corresponding result for a progressive wave.

2. Notation and differential equations

Denote by:

- g the acceleration of gravity,
- h the undisturbed depth of the water, supposed uniform,
- a the length of the estuary,
- x distance down the estuary, $x = 0$ being at the head and $x = a$ at the mouth,
- t the time,
- ζ the elevation of the water-surface,
- u the current down the estuary,
- k a numerical coefficient of friction, which will be taken as 0·0025;

and write

$$c = (gh)^{1/2}.$$

Take also
$$\xi = t+\frac{x}{c}, \qquad \eta = t-\frac{x}{c}, \tag{1}$$
so that
$$t = \tfrac{1}{2}(\xi+\eta), \qquad \frac{x}{c} = \tfrac{1}{2}(\xi-\eta), \tag{2}$$
and
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}, \qquad c\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}-\frac{\partial}{\partial \eta}, \tag{3}$$
so that
$$\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2} = 4\frac{\partial^2}{\partial \xi\,\partial \eta}. \tag{4}$$

Accents will be used to denote derivatives of functions with respect to their arguments. Suffixes 1 and 2 will be used respectively to denote terms which are of the first and second orders in the ratio of the primary surface-elevation to the depth of water.

The equation of continuity is
$$\frac{\partial}{\partial x}\{(h+\zeta)u\}+\frac{\partial \zeta}{\partial t} = 0, \tag{5}$$
and the equation of motion is
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial \zeta}{\partial x} = -\frac{k}{h}|u|u \tag{6}$$
while $u = 0$ where $x = 0$.

The product terms on the left-hand sides of (5), (6) will be called the 'shallow water terms' of these equations.

3. General form of solution

The elimination of ζ from the first-order terms of (5) and (6) gives
$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2} = g\frac{\partial^2}{\partial x^2}(\zeta u)-\frac{1}{2}\frac{\partial^2(u^2)}{\partial x\,\partial t}-\frac{k}{h}\frac{\partial}{\partial t}|u|u, \tag{7}$$
and the general solution of the first-order part of this equation, which makes $u_1 = 0$ where $x = 0$, is
$$u_1/c = -F\left(t+\frac{x}{c}\right)+F\left(t-\frac{x}{c}\right), \tag{8}$$
where $F(\,)$ denotes any function which is physically interpretable. It follows from the first-order part of (5) that
$$\frac{1}{h}\frac{\partial \zeta_1}{\partial t} = -\frac{\partial u_1}{\partial x} = F'\left(t+\frac{x}{c}\right)+F'\left(t-\frac{x}{c}\right),$$
so that
$$\frac{\zeta_1}{h} = F\left(t+\frac{x}{c}\right)+F\left(t-\frac{x}{c}\right), \tag{9}$$
no term independent of t being required.

The terms of (8), (9) in $F(t+x/c)$ are taken to represent the prescribed incident wave. The solution of the fundamental equations, as far as the second order, will be of the form

$$\frac{\zeta}{h} = F\left(t+\frac{x}{c}\right)+F\left(t-\frac{x}{c}\right)+\frac{\zeta_2}{h}, \tag{10}$$

$$\frac{u}{c} = -F\left(t+\frac{x}{c}\right)+F\left(t-\frac{x}{c}\right)+\frac{u_2}{c}. \tag{11}$$

The terms of (10), (11) in $F(t-x/c)$ represent a wave travelling down the estuary, and, to satisfy the condition prescribed in § 1, so must the terms in ζ_2, u_2 at $x = a$. The condition for this is

$$\frac{\zeta_2}{h} = \frac{u_2}{c}, \tag{12}$$

where $x = a$, and this will be used as the determining condition in this paper.

4. Second order terms

For the second approximation, the equations (5), (7) lead to

$$h\frac{\partial u_2}{\partial x} + \frac{\partial \zeta_2}{\partial t} = -\frac{\partial}{\partial x}(\zeta_1 u_1), \tag{13}$$

$$\frac{\partial^2 u_2}{\partial t^2} - c^2\frac{\partial^2 u_2}{\partial x^2} = g\frac{\partial^2}{\partial x^2}(\zeta_1 u_1) - \frac{1}{2}\frac{\partial^2(u_1^2)}{\partial x\,\partial t} - \frac{k}{h}\frac{\partial}{\partial t}|u_1|u_1 \tag{14}$$

respectively, and the transformation (3), (4) applied to (14) gives

$$4\frac{\partial^2 u_2}{\partial\xi\,\partial\eta} = \frac{1}{h}\left(\frac{\partial^2}{\partial\xi^2} - 2\frac{\partial^2}{\partial\xi\,\partial\eta} + \frac{\partial^2}{\partial\eta^2}\right)(\zeta_1 u_1) - \\ - \frac{1}{2c}\left(\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2}\right)u_1^2 - \frac{k}{h}\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)|u_1|u_1. \tag{15}$$

Now (9), (8) may be written respectively as

$$\frac{\zeta_1}{h} = F(\xi)+F(\eta), \tag{16}$$

$$\frac{u_1}{c} = -F(\xi)+F(\eta), \tag{17}$$

and substitution from these into (15) gives

$$\frac{4}{c}\frac{\partial^2 u_2}{\partial\xi\,\partial\eta} = -\left(\frac{\partial^2}{\partial\xi^2} - 2\frac{\partial^2}{\partial\xi\,\partial\eta} + \frac{\partial^2}{\partial\eta^2}\right)\{F^2(\xi) - F^2(\eta)\} - \\ - \frac{1}{2}\left(\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2}\right)\{F(\xi) - F(\eta)\}^2 + \\ + \frac{kc}{h}\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)\{|F(\xi) - F(\eta)|[F(\xi) - F(\eta)]\}. \tag{18}$$

The general solution of (18) is of the form

$$4u_2/c = \phi + \psi + X + Y, \tag{19}$$

where

$$\frac{\partial^2\phi}{\partial\xi\,\partial\eta} = \frac{\partial^2}{\partial\xi^2}\{-\tfrac{3}{2}F^2(\xi) + F(\xi)F(\eta)\} + \frac{kc}{h}\frac{\partial}{\partial\xi}\{|F(\xi) - F(\eta)|[F(\xi) - F(\eta)]\},$$

$$\frac{\partial^2\psi}{\partial\xi\,\partial\eta} = \frac{\partial^2}{\partial\eta^2}\{\tfrac{3}{2}F^2(\eta) - F(\eta)F(\xi)\} + \frac{kc}{h}\frac{\partial}{\partial\eta}\{|F(\xi) - F(\eta)|[F(\xi) - F(\eta)]\},$$

and X, Y are arbitrary functions of ξ, η respectively. One integration of each of these equations gives

$$\frac{\partial\phi}{\partial\eta} = -\tfrac{3}{2}[F^2(\xi)]' + F'(\xi)F(\eta) + \frac{kc}{h}|F(\xi)-F(\eta)|[F(\xi)-F(\eta)],$$

$$\frac{\partial\psi}{\partial\xi} = \tfrac{3}{2}[F^2(\eta)]' - F'(\eta)F(\xi) + \frac{kc}{h}|F(\xi)-F(\eta)|[F(\xi)-F(\eta)],$$

no additional arbitrary functions being required. Further integrations give

$$\phi = \tfrac{3}{2}(\xi-\eta)[F^2(\xi)]' + F'(\xi)\int_\xi^\eta F(\theta)\,d\theta + \\ + \frac{kc}{h}\int_\xi^\eta |F(\xi)-F(\theta)|[F(\xi)-F(\theta)]\,d\theta, \quad (20)$$

$$\psi = \tfrac{3}{2}(\xi-\eta)[F^2(\eta)]' - F'(\eta)\int_\eta^\xi F(\theta)\,d\theta + \\ + \frac{kc}{h}\int_\eta^\xi |F(\theta)-F(\eta)|[F(\theta)-F(\eta)]\,d\theta, \quad (21)$$

the additive functions of ξ and η respectively in these integrals being so chosen that $\phi = \psi = 0$ where $\xi-\eta = 0$.

From (19), (20), (21), and on taking

$$X = -4f(\xi), \qquad Y = 4f(\eta),$$

it follows that

$$\frac{u_2}{c} = \tfrac{3}{8}(\xi-\eta)\{[F^2(\xi)]' + [F^2(\eta)]'\} - \tfrac{1}{4}\{F'(\xi)+F'(\eta)\}\int_\eta^\xi F(\theta)\,d\theta - \\ - \frac{1}{4}\frac{kc}{h}\int_\eta^\xi \{|F(\xi)-F(\theta)|[F(\xi)-F(\theta)] - |F(\theta)-F(\eta)|[F(\theta)-F(\eta)]\}\,d\theta - \\ -f(\xi)+f(\eta). \quad (22)$$

In (22), $u_2 = 0$ where $\xi-\eta = 0$, whatever the function $f(\ )$ may be.

Transformation of (22) back to the independent variables x, t, by using the equations (1), (2), gives

$$\frac{u_2}{c} = \frac{3}{4}\frac{x}{c}\left\{\left[F^2\left(t+\frac{x}{c}\right)\right]' + \left[F^2\left(t-\frac{x}{c}\right)\right]'\right\} - \\ - \frac{1}{4}\left\{F'\left(t+\frac{x}{c}\right) + F'\left(t-\frac{x}{c}\right)\right\}\int_{t-x/c}^{t+x/c} F(\theta)\,d\theta - \\ - \frac{1}{4}\frac{kc}{h}\int_{t-x/c}^{t+x/c}\left\{\left|F\left(t+\frac{x}{c}\right)-F(\theta)\right|\left[F\left(t+\frac{x}{c}\right) - \right.\right. \\ \left.\left. - F(\theta)\right] - \left|F(\theta)-F\left(t-\frac{x}{c}\right)\right|\left[F(\theta)-F\left(t-\frac{x}{c}\right)\right]\right\}d\theta - \\ -f\left(t+\frac{x}{c}\right) + f\left(t-\frac{x}{c}\right); \quad (23)$$

and then substitution from (8), (9), (23) into (13) gives

$$\frac{1}{h}\frac{\partial\zeta_2}{\partial t}=\frac{1}{4}\left\{\left[F^2\left(t+\frac{x}{c}\right)\right]'+\left[F^2\left(t-\frac{x}{c}\right)\right]'\right\}-$$
$$-\frac{3}{4}\frac{x}{c}\left\{\left[F^2\left(t+\frac{x}{c}\right)\right]''-\left[F^2\left(t-\frac{x}{c}\right)\right]''\right\}+$$
$$+\frac{1}{4}\left\{F\left(t+\frac{x}{c}\right)+F\left(t-\frac{x}{c}\right)\right\}\left\{F'\left(t+\frac{x}{c}\right)+F'\left(t-\frac{x}{c}\right)\right\}+$$
$$+\frac{1}{4}\left\{F''\left(t+\frac{x}{c}\right)-F''\left(t-\frac{x}{c}\right)\right\}\int_{t-x/c}^{t+x/c}F(\theta)\,d\theta+$$
$$+\frac{1}{4}\frac{kc^2}{h}\int_{t-x/c}^{t+x/c}\frac{\partial}{\partial x}\left\{\left|F\left(t+\frac{x}{c}\right)-F(\theta)\right|\left[F\left(t+\frac{x}{c}\right)-F(\theta)\right]-\right.$$
$$\left.-\left|F(\theta)-F\left(t-\frac{x}{c}\right)\right|\left[F(\theta)-F\left(t-\frac{x}{c}\right)\right]\right\}d\theta+$$
$$+f'\left(t+\frac{x}{c}\right)+f'\left(t-\frac{x}{c}\right), \quad (24)$$

since the terms due to differentiation with respect to the upper and lower bounds of the integrals in (23) balance. Integration of (24) gives

$$\frac{\zeta_2}{h}=\frac{1}{4}\left\{F\left(t+\frac{x}{c}\right)+F\left(t-\frac{x}{c}\right)\right\}^2-\frac{3}{4}\frac{x}{c}\left\{\left[F^2\left(t+\frac{x}{c}\right)\right]'-\left[F^2\left(t-\frac{x}{c}\right)\right]'\right\}+$$
$$+\frac{1}{4}\left\{F'\left(t+\frac{x}{c}\right)-F'\left(t-\frac{x}{c}\right)\right\}\int_{t-x/c}^{t+x/c}F(\theta)\,d\theta+$$
$$+\frac{1}{4}\frac{kc}{h}\int_{t-x/c}^{t+x/c}\left\{\left|F\left(t+\frac{x}{c}\right)-F(\theta)\right|\left[F\left(t+\frac{x}{c}\right)-F(\theta)\right]+\right.$$
$$\left.+\left|F(\theta)-F\left(t-\frac{x}{c}\right)\right|\left[F(\theta)-F\left(t-\frac{x}{c}\right)\right]\right\}d\theta+$$
$$+f\left(t+\frac{x}{c}\right)+f\left(t-\frac{x}{c}\right). \quad (25)$$

Substitution from (23), (25) into the condition (12) gives

$$2f\left(t+\frac{a}{c}\right)=-\frac{1}{4}\left\{F\left(t+\frac{a}{c}\right)+F\left(t-\frac{a}{c}\right)\right\}^2+$$
$$+F'\left(t+\frac{a}{c}\right)\left\{\frac{3a}{c}F\left(t+\frac{a}{c}\right)-\frac{1}{2}\int_{t-a/c}^{t+a/c}F(\theta)\,d\theta\right\}-$$
$$-\frac{1}{2}\frac{kc}{h}\int_{t-a/c}^{t+a/c}\left|F\left(t+\frac{a}{c}\right)-F(\theta)\right|\left[F\left(t+\frac{a}{c}\right)-F(\theta)\right]d\theta$$

for all values of t, so that

$$f(t)=-\frac{1}{8}\left\{F(t)+F\left(t-\frac{2a}{c}\right)\right\}^2+F'(t)\left\{\frac{3}{2}\frac{a}{c}F(t)-\frac{1}{4}\int_{t-2a/c}^{t}F(\theta)\,d\theta\right\}-$$
$$-\frac{1}{4}\frac{kc}{h}\int_{t-2a/c}^{t}|F(t)-F(\theta)|[F(t)-F(\theta)]\,d\theta. \quad (26)$$

Substitution from (26) into (25) gives

$$\frac{\zeta_2}{h} = \frac{1}{8}F^2\left(t+\frac{x}{c}\right)+\frac{1}{8}F^2\left(t-\frac{x}{c}\right)-\frac{1}{8}F^2\left(t+\frac{x}{c}-\frac{2a}{c}\right)-\frac{1}{8}F^2\left(t-\frac{x}{c}-\frac{2a}{c}\right)+$$
$$+\frac{1}{2}F\left(t+\frac{x}{c}\right)F\left(t-\frac{x}{c}\right)-\frac{1}{4}F\left(t+\frac{x}{c}\right)F\left(t+\frac{x}{c}-\frac{2a}{c}\right)-$$
$$-\frac{1}{4}F\left(t-\frac{x}{c}\right)F\left(t-\frac{x}{c}-\frac{2a}{c}\right)+\frac{3}{2}\frac{a-x}{c}F\left(t+\frac{x}{c}\right)F'\left(t+\frac{x}{c}\right)+$$
$$+\frac{3}{2}\frac{a+x}{c}F\left(t-\frac{x}{c}\right)F'\left(t-\frac{x}{c}\right)-\frac{1}{4}F'\left(t+\frac{x}{c}\right)\int_{t+x/c-2a/c}^{t-x/c}F(\theta)\,d\theta-$$
$$-\frac{1}{4}F'\left(t-\frac{x}{c}\right)\int_{t-x/c-2a/c}^{t+x/c}F(\theta)\,d\theta$$
$$-\frac{1}{4}\frac{kc}{h}\int_{t+x/c-2a/c}^{t-x/c}\left|F\left(t+\frac{x}{c}\right)-F(\theta)\right|\left[F\left(t+\frac{x}{c}\right)-F(\theta)\right]d\theta+$$
$$+\frac{1}{4}\frac{kc}{h}\int_{t-x/c-2a/c}^{t+x/c}\left|F(\theta)-F\left(t-\frac{x}{c}\right)\right|\left[F(\theta)-F\left(t-\frac{x}{c}\right)\right]d\theta, \quad (27)$$

and a similar formula may be obtained for u_2/c.

For the second approximation to be a valid approximation, it is necessary for $|\zeta_2|$ to be small compared with the maximum value of $|\zeta_1|$.

5. Elevation at head of estuary

At the head of the estuary, where $x = 0$, it follows from (10), (27) that

$$\frac{\zeta}{h} = 2F(t)+\tfrac{3}{4}F^2(t)-\tfrac{1}{4}F^2(t-2a/c)-\tfrac{1}{2}F(t)F(t-2a/c)+$$
$$+F'(t)\left\{3\frac{a}{c}F(t)-\frac{1}{2}\int_{t-2a/c}^{t}F(\theta)\,d\theta\right\}-$$
$$-\frac{1}{2}\frac{kc}{h}\int_{t-2a/c}^{t}|F(t)-F(\theta)|[F(t)-F(\theta)]\,d\theta. \quad (28)$$

It is then seen that the elevation of the water-surface at the head of the estuary depends upon the first-order elevation there at the same time and for a previous interval during which a first-order progressive wave could travel twice the length of the estuary.

When the first-order elevation at the head of the estuary has a positive maximum at $t = 0$, then $F'(0) = 0$ and $F''(0)$ is negative. The formula (28) then gives, for that time,

$$\frac{\zeta}{h} = 2F(0)+\tfrac{3}{4}F^2(0)-\tfrac{1}{4}F^2(-2a/c)-\tfrac{1}{2}F(0)F(-2a/c)-$$
$$-\frac{1}{2}\frac{kc}{h}\int_{-2a/c}^{0}|F(0)-F(\theta)|[F(0)-F(\theta)]\,d\theta, \quad (29)$$

$$\frac{1}{h}\frac{\partial\zeta}{\partial t} = -\tfrac{1}{2}F'(-2a/c)\{F(0)+F(-2a/c)\}+$$
$$+F''(0)\left\{\frac{3a}{c}F(0)-\frac{1}{2}\int_{-2a/c}^{0}F(\theta)\,d\theta\right\}+$$
$$+\frac{1}{2}\frac{kc}{h}|F(0)-F(-2a/c)|\{F(0)-F(-2a/c)\}, \quad (30)$$

and (30) is of the second order in F.

Now
$$\frac{\partial\zeta}{\partial t} = \left(\frac{\partial\zeta}{\partial t}\right)_{t=0} + t\left(\frac{\partial^2\zeta}{\partial t^2}\right)_{t=0},$$
to the first order in t, and, as high water occurs when $\partial\zeta/\partial t = 0$, it follows from (30) that the time of high water is given, to the first order in F, by
$$t = -\frac{3}{2}\frac{a}{c}F(0) + \frac{1}{4}\frac{F'(-2a/c)}{F''(0)}\{F(0)+F(-2a/c)\} + \frac{1}{4}\int_{-2a/c}^{0} F(\theta)\,d\theta - \\ -\frac{1}{4}\frac{kc}{h}\frac{1}{F''(0)}|F(0)-F(-2a/c)|\{F(0)-F(-2a/c)\}. \quad (31)$$
Also,
$$\zeta = (\zeta)_{t=0} + t\left(\frac{\partial\zeta}{\partial t}\right)_{t=0},$$
so that, to the second order in F, the height of high water is given by (29).

Suppose now that
$$0 < F(-2a/c) < F(\theta) < F(0), \quad \text{for } -2a/c < \theta < 0,$$
so that $F'(-2a/c)$ is positive. Then from (29), (31) it can be shown that the tendency of the shallow water terms of the fundamental equations is to make high water higher and earlier, and that the tendency of the frictional term is to make high water lower and later.

When there is no surge and the first-order tide is harmonic of amplitude $2hA$ and period $2\pi/\sigma$ so that
$$F(t) = A\cos\sigma t,$$
then the formulae (29), (31) for the height and time of high water at the head of the estuary become respectively
$$\frac{\zeta}{h} = 2A + A^2\left\{\frac{5}{8} - \frac{1}{2}\cos\frac{2\sigma a}{c} - \frac{1}{8}\cos\frac{4\sigma a}{c} - \\ -\frac{1}{2}\frac{kc}{h\sigma}\left(\frac{3\sigma a}{c} - 2\sin\frac{2\sigma a}{c} + \frac{1}{4}\sin\frac{4\sigma a}{c}\right)\right\},$$
$$t = \frac{A}{\sigma}\left\{-\frac{3}{2}\frac{\sigma a}{c} - \frac{1}{8}\sin\frac{4\sigma a}{c} + \frac{1}{8}\frac{kc}{h\sigma}\left(3 - 4\cos\frac{2\sigma a}{c} + \cos\frac{4\sigma a}{c}\right)\right\}.$$

In the tidal example taken by Doodson, $h = 128$ ft, $2\pi/\sigma = 12$ hours, $\sigma a/c = 2\pi/5$, $2hA = 5$ ft, so that $A = 0{\cdot}0195$, $kc/h\sigma = 6{\cdot}9$, $\frac{1}{4}Akc/h\sigma = 0{\cdot}034$. These figures indicate that the method of the present paper should give an approximation to the motion. But for an estuary, the depths are smaller, so that the values of $kc/h\sigma$ are larger, and the values of A may be much larger. In these circumstances the approximation will only be valid if $\sigma a/c$ be small.

6. Short estuary

Now suppose that, during the time taken by a progressive wave to travel the length of the estuary, the elevation at any one place changes by only a small fraction of its maximum value. Then aF'/c will be small compared with the maximum value of F.

On expanding the second-order terms in ascending powers of x/c as far as x^3/c^3, it follows that

$$\int_{t-x/c}^{t+x/c} F(\theta)\,d\theta = \int_{t-x/c}^{t+x/c} \{F(t)+(\theta-t)F'(t)+\tfrac{1}{2}(\theta-t)^2F''(t)\}\,d\theta$$
$$= 2\frac{x}{c}F(t)+\frac{1}{3}\frac{x^3}{c^3}F''(t),$$

and similarly it may be shown that

$$\int_{t-2a/c}^{t} F(\theta)\,d\theta = \frac{2a}{c}F(t)-\frac{2a^2}{c^2}F'(t)+\frac{4}{3}\frac{a^3}{c^3}F''(t).$$

Also

$$\int_{t-x/c}^{t+x/c} |F(t+x/c)-F(\theta)|[F(t+x/c)-F(\theta)]\,d\theta$$
$$= |F'(t)|F'(t)\int_{t-x/c}^{t+x/c}(t+x/c-\theta)^2\,d\theta$$
$$= \frac{8}{3}\frac{x^3}{c^3}|F'(t)|F'(t),$$

and similarly it may be shown that

$$\int_{t-x/c}^{t+x/c} |F(\theta)-F(t-x/c)|[F(\theta)-F(t-x/c)]\,d\theta = \frac{8}{3}\frac{x^3}{c^3}|F'(t)|F'(t),$$
$$\int_{t-2a/c}^{t} |F(t)-F(\theta)|[F(t)-F(\theta)]\,d\theta = \frac{8}{3}\frac{a^3}{c^3}|F'(t)|F'(t).$$

It then follows, from (25), (23), (26) respectively, and on omitting the argument t of F and f, that

$$\frac{\zeta_2}{h} = F^2-\frac{x^2}{c^2}(FF''+3F'^2)+\frac{4}{3}\frac{kx^3}{hc^2}|F'|F'+2f+\frac{x^2}{c^2}f'', \quad (32)$$

$$\frac{u_2}{c} = 2\frac{x}{c}FF'+\frac{x^3}{c^3}\left(FF'''+\frac{13}{3}F'F''\right)-2\frac{x}{c}f'-\frac{1}{3}\frac{x^3}{c^3}f''', \quad (33)$$

$$f = -\tfrac{1}{2}F^2+\frac{2a}{c}FF'-\frac{a^2}{c^2}FF''+\frac{2}{3}\frac{a^3}{c^3}(FF'''+F'F'')-\frac{2}{3}\frac{ka^3}{hc^2}|F'|F'. \quad (34)$$

From (10), (32), (33), (34), it then follows that

$$\frac{\zeta}{h} = F\left(t+\frac{x}{c}\right)+F\left(t-\frac{x}{c}\right)+\frac{4a}{c}FF'-\frac{2a^2}{c^2}FF''-\frac{2x^2}{c^2}(FF''+2F'^2)+$$
$$+\frac{4}{3}\frac{a^3}{c^3}(FF'''+F'F'')+\frac{2ax^2}{c^3}(FF'''+3F'F'')-$$
$$-\frac{4}{3}\frac{k}{hc^2}(a^3-x^3)|F'|F', \quad (35)$$

$$\frac{u}{c} = -F\left(t+\frac{x}{c}\right)+F\left(t-\frac{x}{c}\right)+\frac{4x}{c}FF'-\frac{4ax}{c^2}(FF''+F'^2)+$$
$$+\frac{2a^2x}{c^3}(FF'''+F'F'')+\frac{4}{3}\frac{x^3}{c^3}(FF'''+4F'F''), \quad (36)$$

the argument of F being t, except where otherwise shown.

It will be seen that, to the order x^3/c^3, the current is independent of friction.

When $F(0) > 0, \quad F'(0) = 0, \quad F''(0) < 0,$

the formulae (29), (31), for the approximate height and time of high water at the head of the estuary, become

$$\frac{\zeta}{h} = 2F(0)\left\{1 - \frac{a^2}{c^2}F''(0) + \frac{2}{3}\frac{a^3}{c^3}F'''(0)\right\}, \tag{37}$$

$$t = -\frac{2a}{c}F(0) + \frac{a^2}{c^2}\frac{F(0)}{F''(0)}\left\{F'''(0) - \frac{2}{3}\frac{a}{c}F^{\text{iv}}(0)\right\} - \frac{2}{3}\frac{a^3}{c^3}F''(0). \tag{38}$$

Both these results are independent of friction.

When again $F(t) = A\cos\sigma t$

A, σ being constants, the formulae (35), (36) become

$$\frac{\zeta}{h} = 2A\cos\frac{\sigma x}{c}\cos\sigma t + A^2\left\{\left[-\frac{2\sigma a}{c} + \frac{4\sigma^3}{c^3}(\tfrac{1}{3}a^3 + ax^2)\right]\sin 2\sigma t + \frac{\sigma^2}{c^2}(a^2 - x^2) + \frac{\sigma^2}{c^2}(a^2 + 3x^2)\cos 2\sigma t + \frac{4}{3}\frac{k\sigma^2}{hc^2}(a^3 - x^3)|\sin\sigma t|\sin\sigma t\right\},$$

$$\frac{u}{c} = 2A\sin\frac{\sigma x}{c}\sin\sigma t + A^2\left\{\left[-\frac{2\sigma x}{c} + \frac{2\sigma^3}{c^3}\left(a^2x + \frac{5}{3}x^3\right)\right]\sin 2\sigma t + \frac{4\sigma^2 ax}{c^2}\cos 2\sigma t\right\},$$

respectively. The formulae (37), (38), for the height and time of high water at the head of the estuary, become

$$\frac{\zeta}{h} = 2A\left(1 + \frac{\sigma^2a^2}{c^2}A^2\right),$$

$$t = -\frac{2a}{c}A\left(1 - \frac{2}{3}\frac{\sigma^2a^2}{c^2}\right),$$

respectively.

The validity of the expansions in x/c requires that $\sigma a/c < 1$. For the above formulae to give a valid approximation to a solution of the fundamental equations, it is necessary for $(\sigma a/c)A$, $(k\sigma^2a^3/hc^2)A$ to be small.

As an example, take h = 40 ft and a semi-diurnal tide. Then c = 24·5 miles per hour, $\sigma = 2\pi/12$ hours^{-1}, so that c/σ = 47 miles. With a = 20 miles and $A = \frac{1}{3}$, this gives

$$\frac{\sigma a}{c}A = 0{\cdot}14, \qquad \frac{k\sigma^2a^3}{hc^2}A = 0{\cdot}39.$$

7. Separation of tide and surge

Take
$$F(t) = T(t) + S(t), \tag{39}$$

where $T(\,)$ denotes a tide and $S(\,)$ a surge. Equal sequences of meteorological conditions over the sea may be regarded as giving rise to equal functions $S(t)$. Since the timing of the meteorological conditions which generate surges is independent of the timing of the astronomical forces which generate the tides, the frequency-distribution of the first-order surges S will be independent of the phases of the first order tides T.

Substitution from (39) into (35) with $x = 0$, gives for the height of the combination of tide and surge at the head of the estuary

$$\frac{\zeta}{h} = 2(T+S) + \frac{4a}{c}(T+S)(T'+S') - \frac{2a^2}{c^2}(T+S)(T''+S'') + \\ + \frac{4}{3}\frac{a^3}{c^3}\{(T+S)(T'''+S''') + (T'+S')(T''+S'')\} - \\ - \frac{4}{3}\frac{ka^3}{hc^2}|T'+S'|(T'+S'), \quad (40)$$

the argument of T and S being t.

The predicted tide would be given by

$$\frac{\zeta}{h} = 2T + \frac{4a}{c}TT' - \frac{2a^2}{c^2}TT'' + \frac{4}{3}\frac{a^3}{c^3}(TT''' + T'T'') - \frac{4}{3}\frac{ka^3}{hc^2}|T'|T', \quad (41)$$

so that the apparent surge is given by subtracting (41) from (40). The formula for the apparent surge is thus

$$\frac{\zeta}{h} = 2S + \frac{4a}{c}SS' - \frac{2a^2}{c^2}SS'' + \frac{4}{3}\frac{a^3}{c^3}(SS''' + S'S'') + \frac{4a}{c}(TS' + T'S) - \\ - \frac{2a^2}{c^2}(TS'' + T''S) + \frac{4}{3}\frac{a^3}{c^3}(TS''' + T'''S + T'S'' + T''S' - \\ - \frac{4}{3}\frac{ka^3}{hc^2}\{|T'+S'|(T'+S') - |T'|T'\}. \quad (42)$$

When the first-order surge has either a maximum or a minimum so that $S' = 0$, the formula (42) reduces to

$$\frac{\zeta}{h} = 2S\left(1 - \frac{a^2}{c^2}S'' + \frac{2}{3}\frac{a^3}{c^3}S''' + \frac{2a}{c}T' - \frac{a^2}{c^2}T'' + \frac{2}{3}\frac{a^3}{c^3}T'''\right) - \\ - \frac{2a^2}{c^2}TS'' + \frac{4}{3}\frac{a^3}{c^3}(TS''' + T'S''), \quad (43)$$

and this is free from frictional influence.

When the terms in a^3/c^3 are negligible, and, at the same time, the first-order tide is either at high water or low water, so that $T' = 0$, the formula (43), for the apparent surge at the head of the estuary, reduces to

$$\frac{\zeta}{h} = 2S\left(1 - \frac{a^2}{c^2}S''\right) - \frac{2a^2}{c^2}(T''S + TS''). \quad (44)$$

Now suppose that

$$S > 0, \qquad S'' < 0,$$

so that the first-order surge has a positive maximum. When the first-order tide is at high water,

$$T = A, \qquad T'' = -\sigma^2 A,$$

hA denoting the amplitude of the tide and $2\pi/\sigma$ its period, and the apparent surge (44) is given by

$$\frac{\zeta}{h} = 2S\left(1-\frac{a^2}{c^2}S''\right)+\frac{2a^2}{c^2}A(\sigma^2S-S''). \tag{45}$$

When the first-order tide is at low water,

$$T = -A, \qquad T'' = \sigma^2 A,$$

and the apparent surge (44) is given by

$$\frac{\zeta}{h} = 2S\left(1-\frac{a^2}{c^2}S''\right)-\frac{2a^2}{c^2}A(\sigma^2S-S''). \tag{46}$$

The excess of (45) over (46) is

$$\frac{4a^2}{c^2}A(\sigma^2S-S''), \tag{47}$$

and this is positive.

It is thus seen that, at the head of a very short estuary, the effect of the interaction of the tide on the surge is to make the apparent surge higher when its maximum occurs at the time of tidal high water than when its maximum occurs at the time of tidal low water. This result, which is due to the shallow water terms of the differential equations, is directly opposite to the corresponding result for a progressive wave. That result, in the case of an estuary of uniform cross-section, is due to the frictional term.

To examine the order of magnitude of the difference (47), take the particular case in which, when the first-order surge is at its maximum, $S'' = -\sigma^2 S$. The difference in the heights of the apparent surges is then equal to the height of the first-order surge multiplied by $(8\sigma^2a^2/c^2)A$. With the figures at the end of § 6, this factor is 0·47.

References

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